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DECOUPLING THE INTEGRALS OF COSMOLOGICAL PERTURBATION THEORY

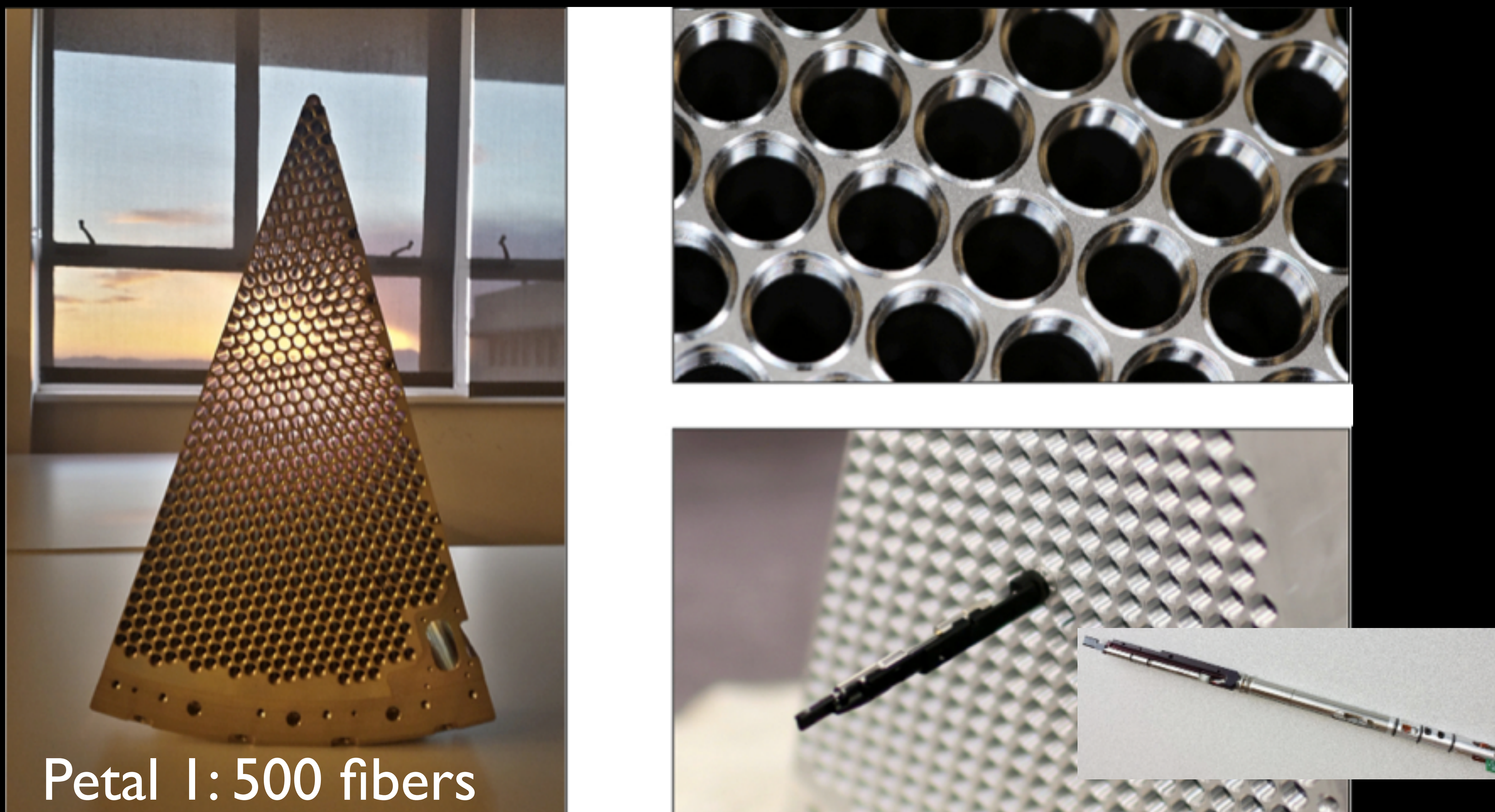
WORK WITH **BOB CAHN**

EINSTEIN SYMPOSIUM | CFA | CAMBRIDGE | 2 OCTOBER 2018


UF
Department of Astronomy
College of Liberal Arts and Sciences

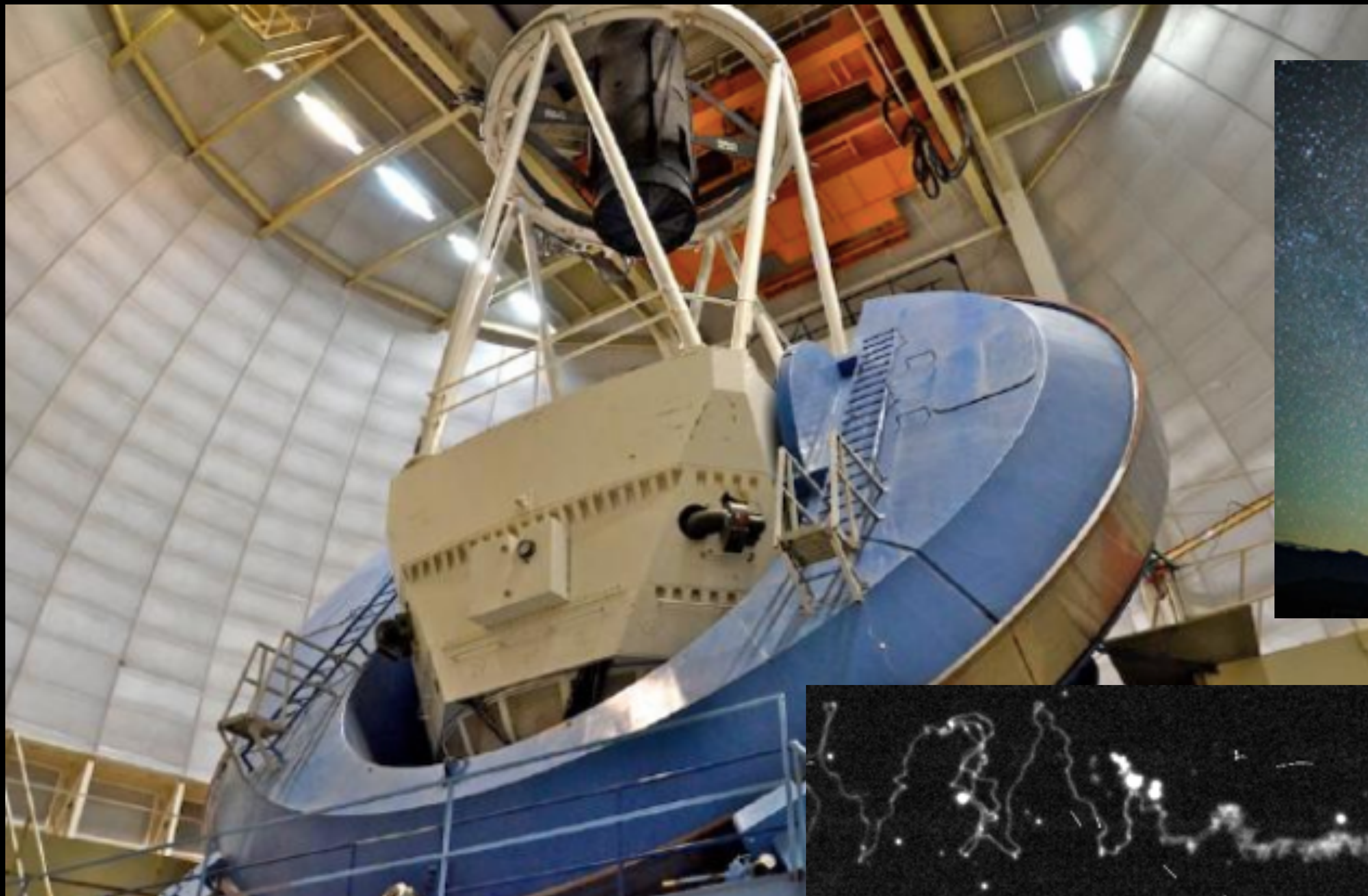
SDSS BOSS: ~1 M galaxies (to 2016)

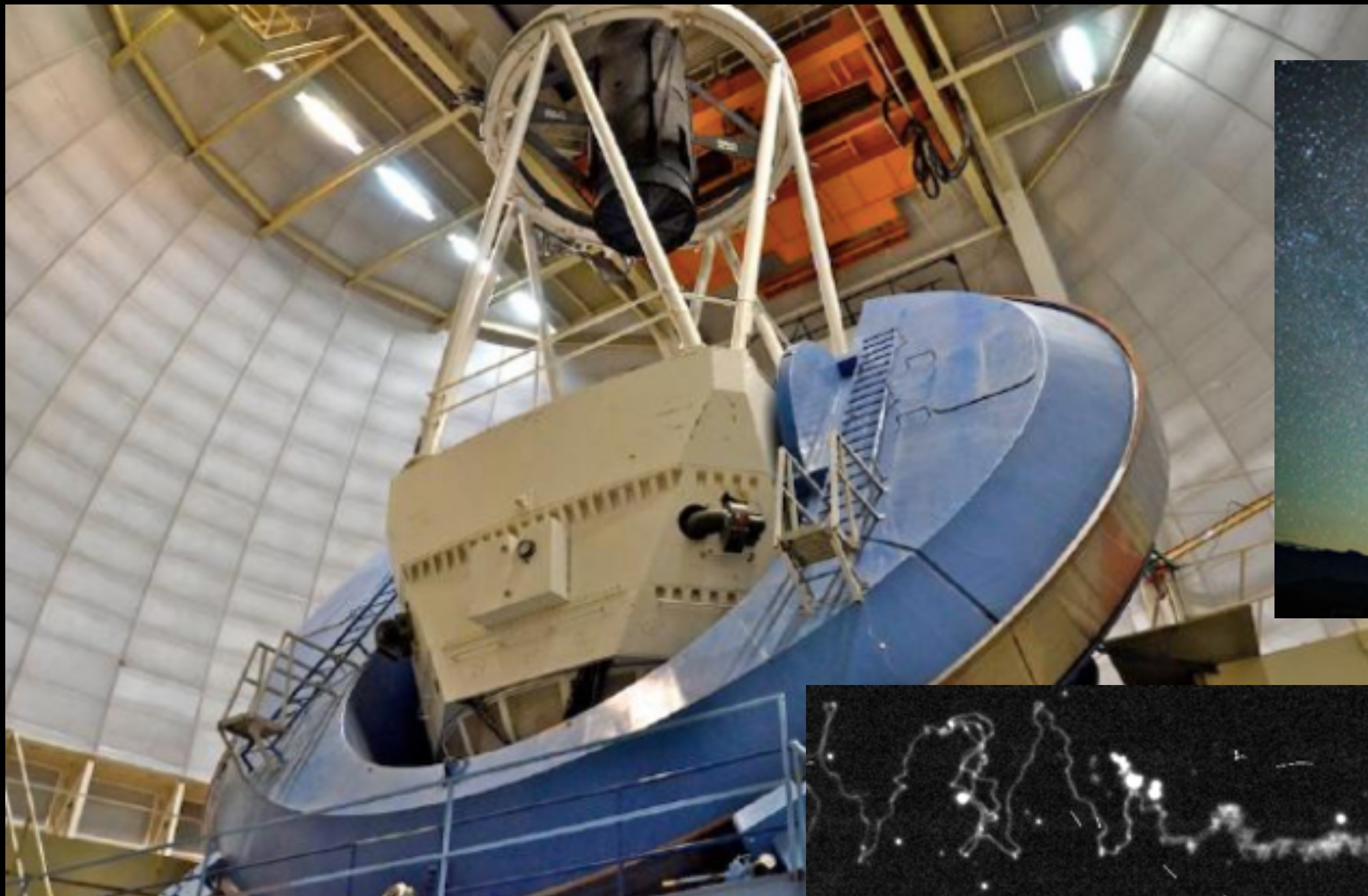
DESI: ~30 M galaxies, quasars (2019-2024)

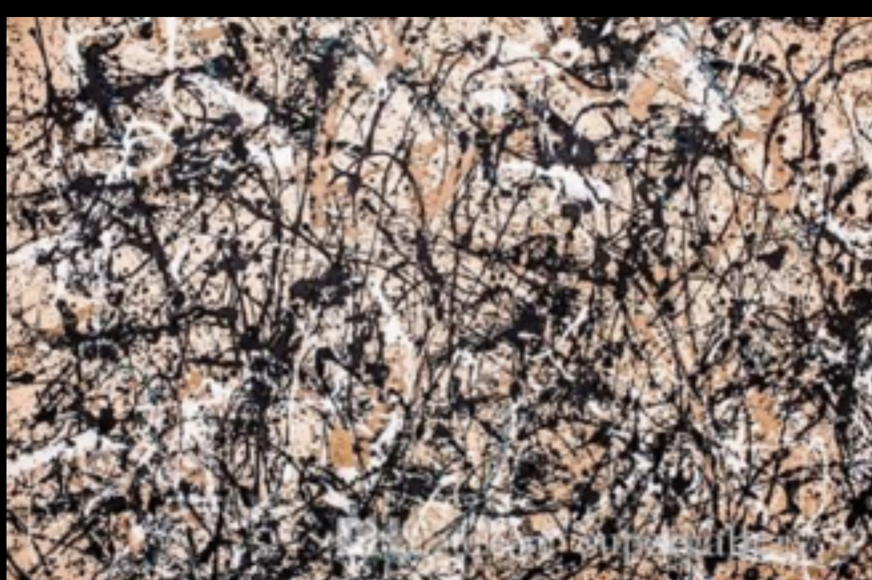


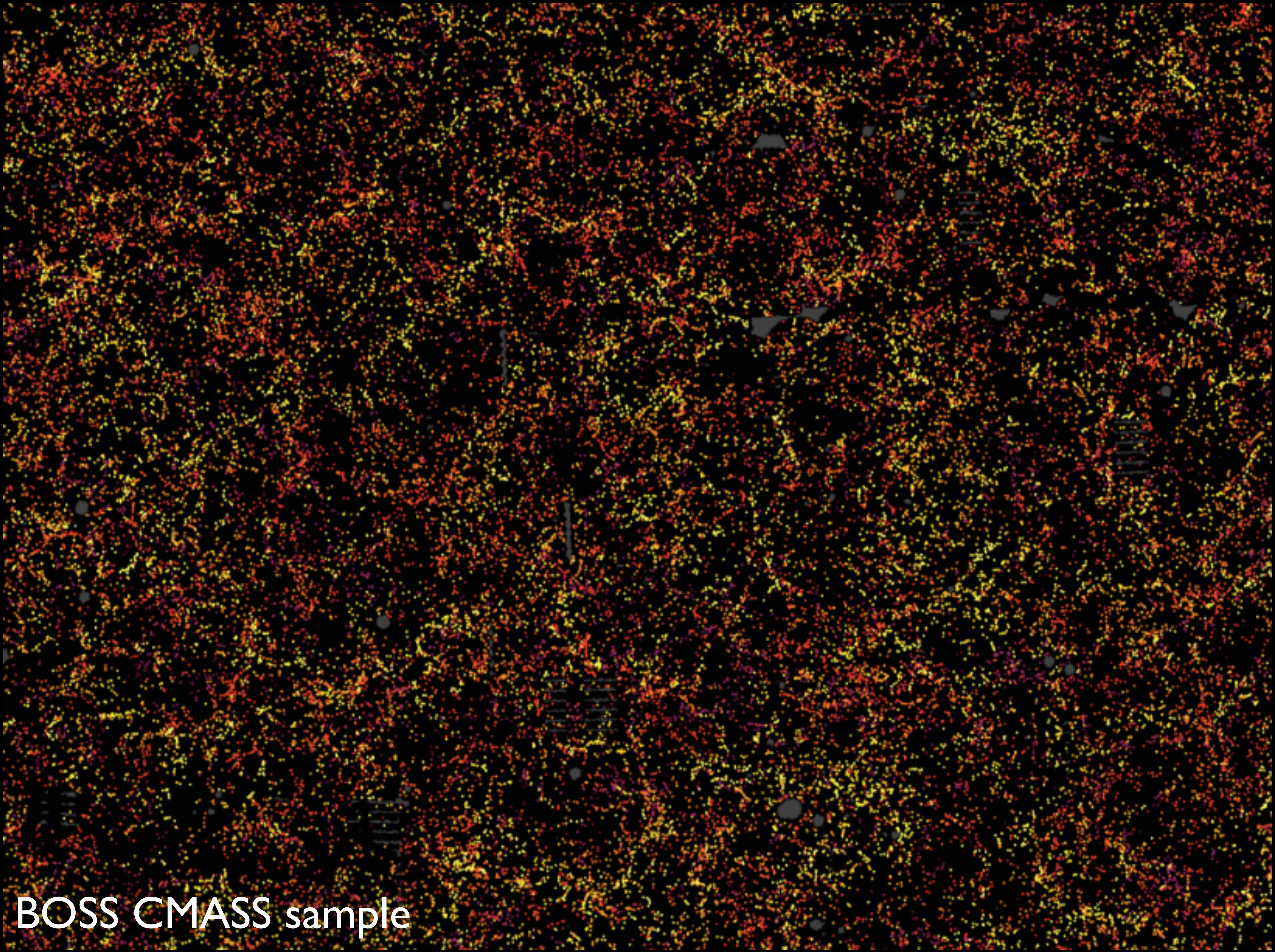
5k robotic positioners: reconfigure in ~1 minute

http://www.symmetrymagazine.org/sites/default/files/images/standard/proto10mm_pos.jpg








BOSS CMASS sample

Ultimate goal: *cosmological parameters*

6 numbers, perhaps 8

Density of matter, baryons, & dark energy; dark energy w, H_0, n_s, σ_8; m_ν, r

Test gravity and inflation

To hold the Universe's origins and eventual fate all at once together in the mind

Initial conditions:
end of inflation

$P_{\rm pri} = Ak^{ns}$

Linear Theory:
each *k* evolves indptly.
CMBFAST/CAMB/CLASS

$P_{\rm lin} = T_{\rm m}^2(k, z)P_{\rm pri}$

$z \sim 100$

Non-Linear Evolution:
-N-Body
-Cosmological Perturbation Theory

$z \sim 0\text{-}2$

Low-z surveys:
DESI, BOSS

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Linear Theory: each k evolves indptly. CMBFAST/ CAMB/CLASS

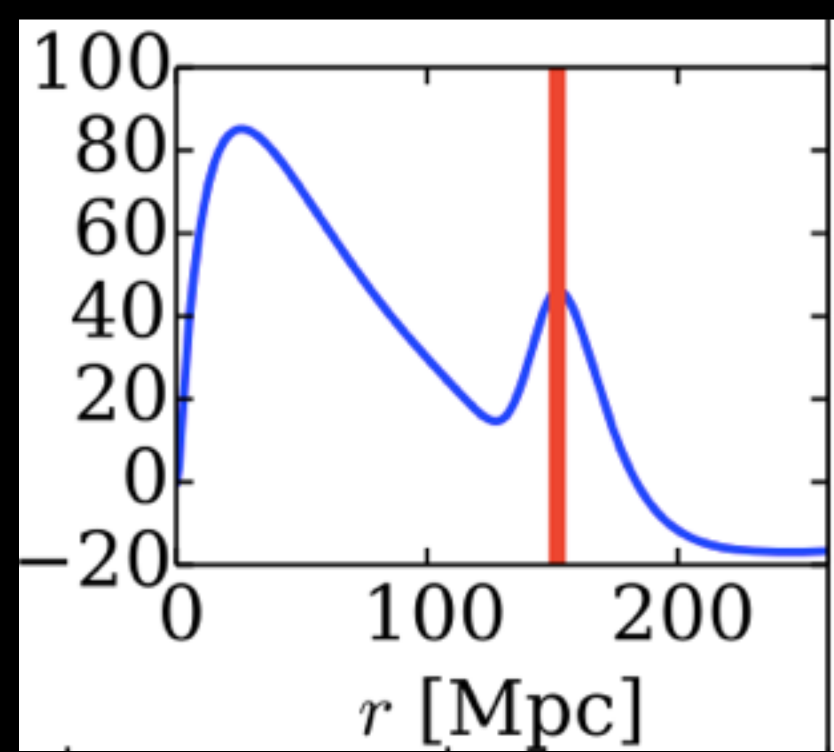
$P_{\rm lin} = T_{\rm m}^2(k, z) P_{\rm pri}$

z ~ 100

Non-Linear Evolution: -N-Body -Cosmological Perturbation Theory

z ~ 0-2

Correlation function/ power spectrum



Initial conditions: end of inflation

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Linear Theory: each *k* evolves indptly. CMBFAST/ CAMB/CLASS

$P_{\rm lin} = T_{\rm m}^2(k, z)P_{\rm pri}$

z ~ 100

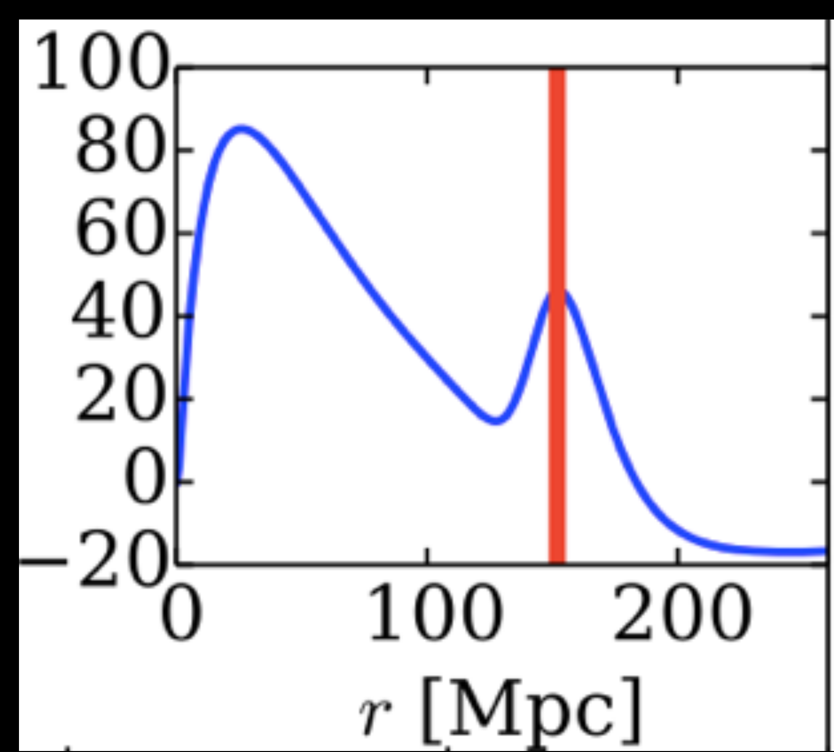
Non-Linear Evolution: -N-Body -Cosmological Perturbation Theory

z ~ 0-2

homogeneous

isotropic

meaning of law



Correlation function/ power spectrum

Perturbation theory: observed statistics of galaxy clustering → linear theory → cosmological parameters

Analyze DESI: MCMC over millions of cosmo. parameter sets → observable stats.

Can we make PT faster?

COSMOLOGICAL PERTURBATION THEORY

Focus on CDM, assume fluid, consider solely gravity

Mass conservation, momentum conservation, gravitational potential

$$\dot{\delta} + \nabla \cdot [(1+\delta)\vec{v}] = 0$$

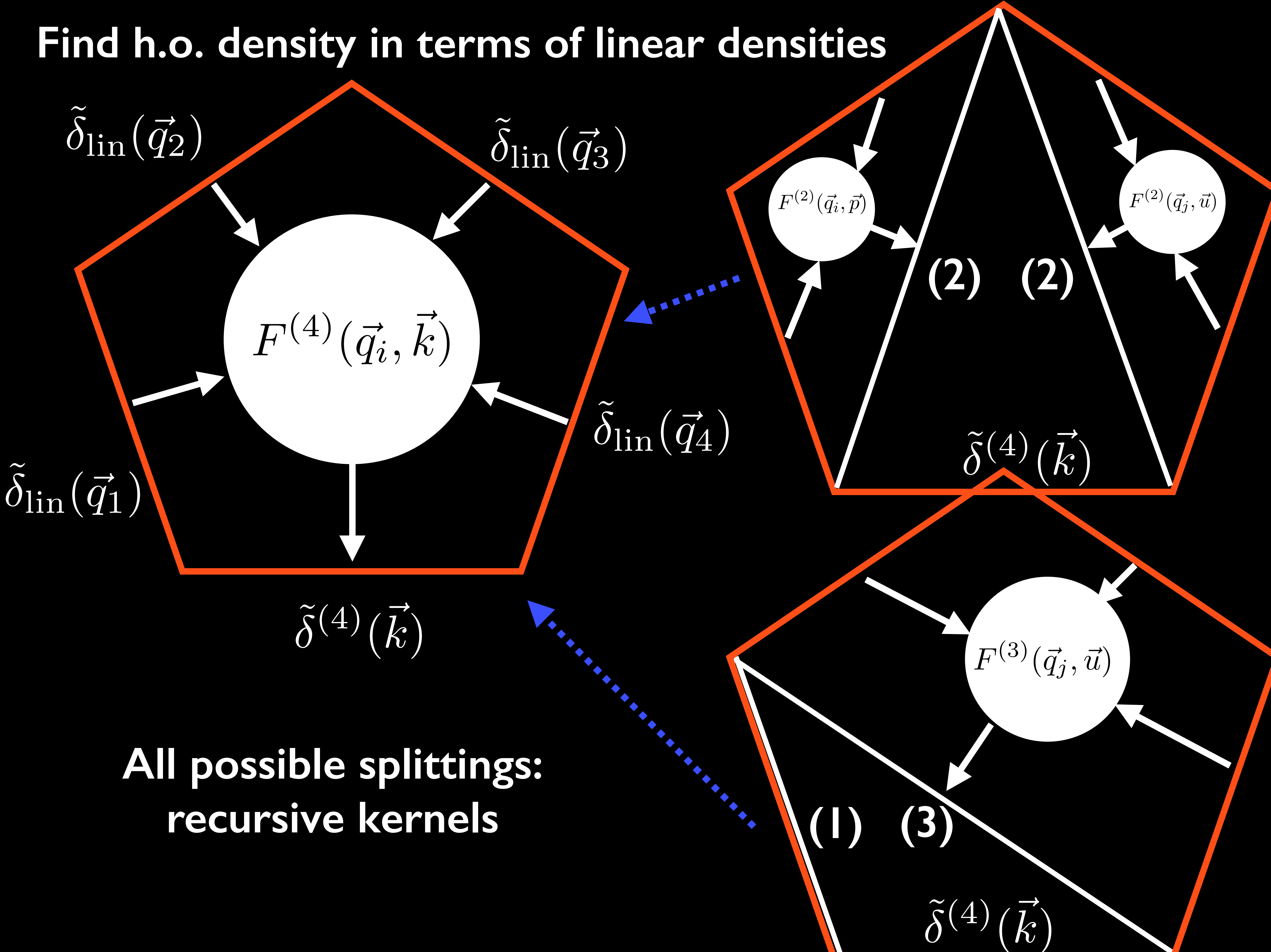
$$\dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v} = -\mathcal{H}\vec{v} - \nabla\phi$$

$$\nabla^2\phi = 4\pi G\bar{\rho}a^2\delta$$

Drop non-linear terms ➔ linear solution
Non-linear terms ➔ mode coupling

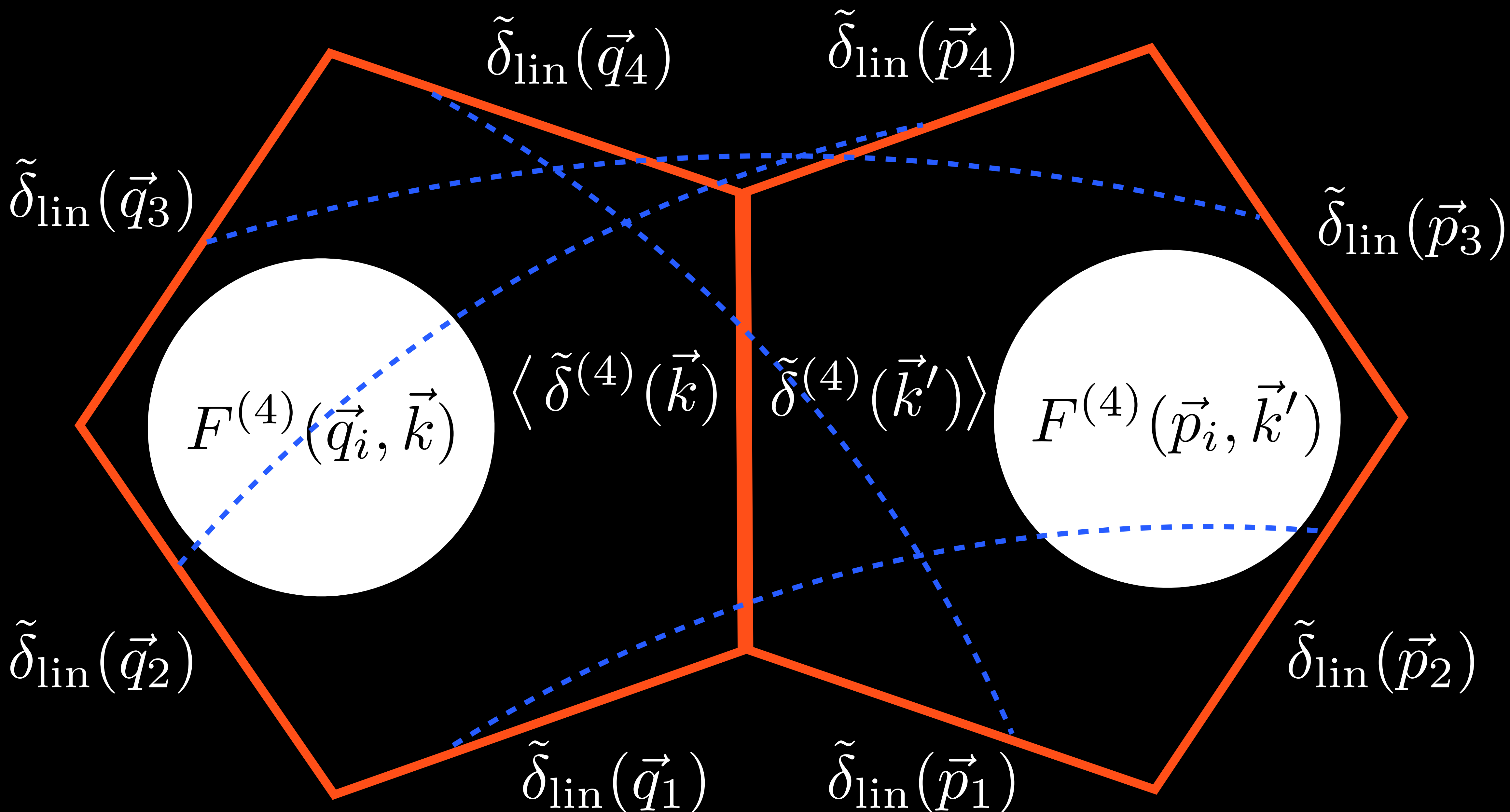
Write in Fourier space, get recursion relation for kernels to integrate against to generate h.o. solutions from lower

Find h.o. density in terms of linear densities



All possible splittings: recursive kernels

Get non-linear density statistics in terms of those of linear density



All possible pairwise gluings of linear densities: “contractions”
→ linear power spectra

FOR INSTANCE

$$P_{\rm non-lin,1-loop}(k) = P_{\rm lin}(k) + 2P_{13}(k) + P_{22}(k)$$

$$P_{\rm non-lin,2-loop}(k) = P_{\rm non-lin,1-loop}(k)$$

$$+P_{15}(k) + 2P_{24}(k) + P_{33}(k)$$

GIVES HI-D COUPLED INTEGRALS

$$I_{ij}(k) = \sum_{\ell m} \frac{4\pi}{2\ell+1} \int \frac{d\Omega_k}{4\pi} \int \frac{d^3\vec{q}_1}{(2\pi)^3} \frac{d^3\vec{q}_2}{(2\pi)^3}$$

$$\times \frac{N_\ell^{[1]}(q_1) N_\ell^{[2]}(q_2) Y_{\ell m}(\hat{q}_1) Y^*_{\ell m}(\hat{q}_1)}{q_1^{2n_1} |\vec{k}+\vec{q}_1|^{2n_1'} q_2^{2n_2} |\vec{k}+\vec{q}_2|^{2n_2'}}$$

$$\times \frac{P_{\rm lin}(q_1) P_{\rm lin}(q_2)}{|\vec{q}_1+\vec{q}_2|^{2n_3} |\vec{k}+\vec{q}_1+\vec{q}_2|^{2n_3'}} P_{\rm lin}(|\vec{w}_{ij}|)$$

2-loop → 6th order: 15, 24, 33

$$\vec{w}_{15} = \vec{k}, \quad \vec{w}_{24} = \vec{k}+\vec{q}_2, \quad \vec{w}_{33} = \vec{k}+\vec{q}_1+\vec{q}_2.$$

Super computationally costly to compute loop corrections
Denominators come from inverse nablas evaluated at non-linear momenta

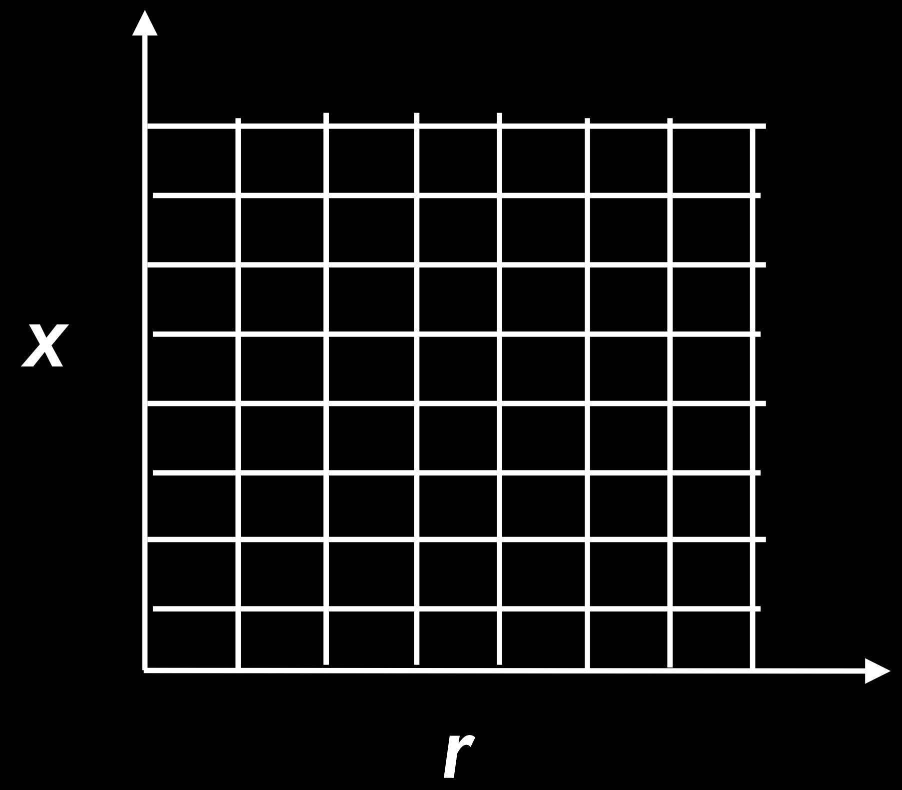
CAN WE . . .

Evaluate this coupled 9-D integral as a series of nested 3-D convolutions?

WHY CONVOLUTIONS?

$$[f \star g](\vec{r}) = \int d^3\vec{x}\ f(\vec{x})g(\vec{x}+\vec{r})$$

Looks like a 3-D integral at every 3-D vector *r*: N^2

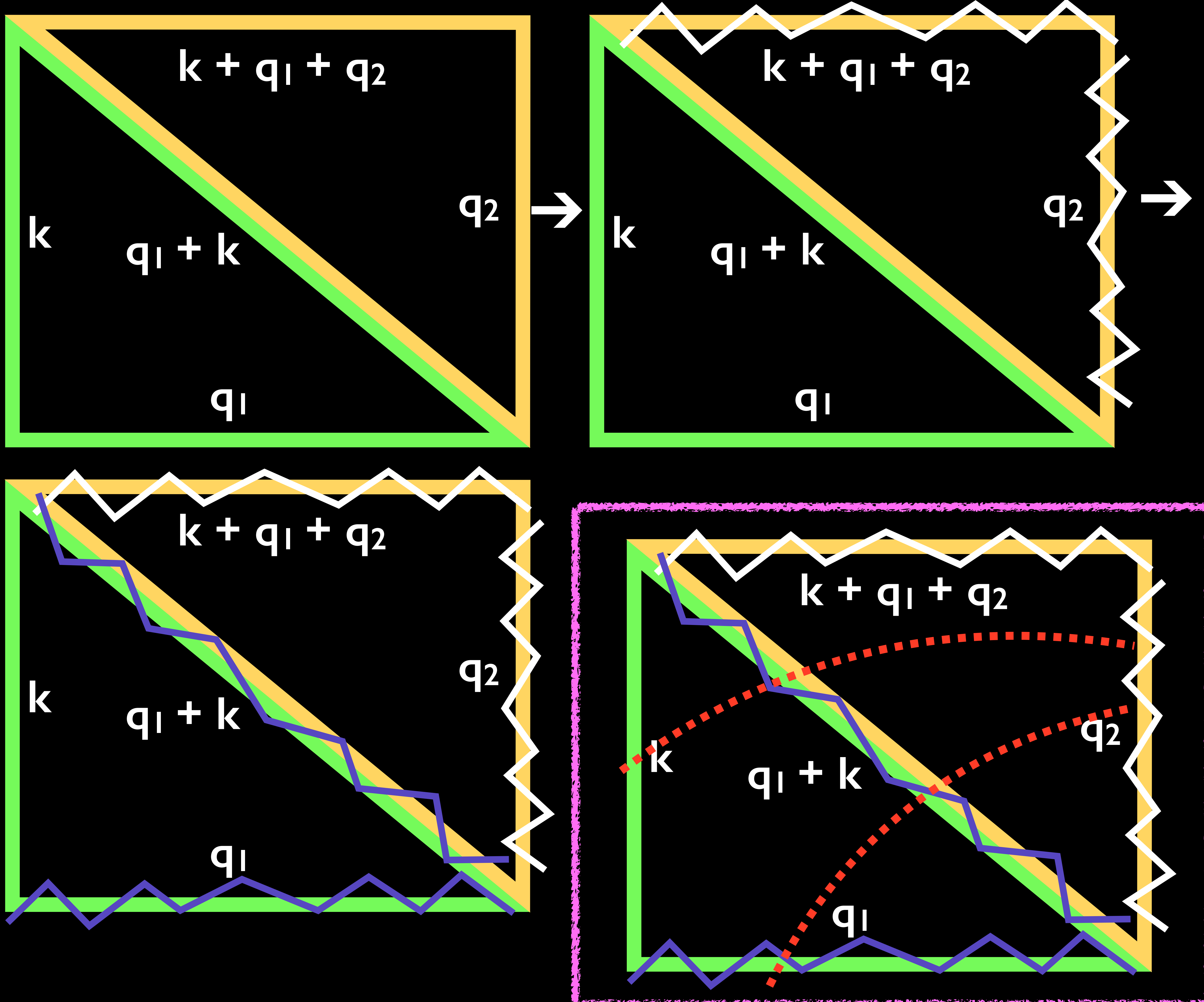


Convolution Theorem: do this as a product in Fourier space, becomes *N log N* instead of N^2

FURTHER. . .

Can we exploit the fact that the power spectrum is the only thing *not known* analytically, and it is isotropic, to . . .

reduce the 3-D convolutions to 1-D ones, to be done with FFTs?



$k + q_1 + q_2$
q_2
k
$q_1 + k$
q_1
$k + q_1 + q_2$
q_2
k
$q_1 + k$
q_1
$k + q_1 + q_2$
q_2
k
$q_1 + k$
q_1
$k + q_1 + q_2$
q_2
k
$q_1 + k$
q_1

$$I_{ij}(k) = \sum_{\ell m} \frac{4\pi}{2\ell+1} \int \frac{d\Omega_k}{4\pi} \int \frac{d^3\vec{q}_1}{(2\pi)^3} \frac{d^3\vec{q}_2}{(2\pi)^3}$$

$$\times \frac{N_\ell^{[1]}(q_1) N_\ell^{[2]}(q_2) Y_{\ell m}(\hat{q}_1) Y^*_{\ell m}(\hat{q}_1)}{q_1^{2n_1} |\vec{k}+\vec{q}_1|^{2n'_1} q_2^{2n_2} \boxed{|\vec{k}+\vec{q}_2|^{2n'_2}}}$$

$$\times \frac{P_{\rm lin}(q_1) P_{\rm lin}(q_2)}{\boxed{\boxed{|\vec{q}_1+\vec{q}_2|^{2n_3}} |\vec{k}+\vec{q}_1+\vec{q}_2|^{2n'_3}}} P_{\rm lin}(|\vec{w}_{ij}|)$$

$$\vec{w}_{15} = \vec{k}, \quad \vec{w}_{24} = \vec{k}+\vec{q}_2, \quad \vec{w}_{33} = \vec{k}+\vec{q}_1+\vec{q}_2.$$

Inner: **15: group q_1 and k and convolve over q_2 in q_2, $q_2 + (k + q_1)$**

24: convolve over q_2 in q_2, $q_2 + k$

33: same as 15

Problem: coupled denominators, boxed red for 15/33, blue for 24

HOW DO WE FACTOR DENOMINATORS?

$$\frac{1}{|\vec{p}_1 + \vec{p}_2|^N}, \quad N = 2, 4$$

If N =1 could use a multipole expansion
Generalization is Gegenbauer expansion

$$\frac{1}{|\vec{p}_1 + \vec{p}_2|^{2\lambda}} = \sum_{\ell=0}^{\infty} \frac{p_<^{\ell}}{p_>^{\ell+2\lambda}} C_\ell^{(\lambda)}(\hat{p}_1 \cdot \hat{p}_2)$$

Use Gegenbauer polynomial addition theorem to separate into f(p_1)g(p_2), turns out better to split into spherical harmonics using “mixed” addition theorem

But still a problem: radial term is only *formally* factored: have constraint $p_1 < p_2$ or visa versa: “1/2-plane constraint”

USING A DECOUPLING INTEGRAL

$$\frac{2}{\pi}\int_0^\infty dx\, x\left[j_{\ell+1}(xp_1)j_\ell(xp_2)+j_{\ell+1}(xp_2)j_\ell(xp_1)\right]$$

$$=\frac{p_2^\ell}{p_1^{\ell+2}},\ \ p_1>p_2,\ \ \frac{p_1^\ell}{p_2^{\ell+2}},\ \ p_2>p_1$$

Can prove using $j_L j_L$ expansion for $1/|p_1 + p_2|$, comparing that with multipole expansion, and then using recursion relations for sBFs

This integral *truly* factorizes the problem, as it always enforces the "1/2-plane constraint"

Can now integrate over momentum magnitudes separately

But the price is an extra integral over *x* at the end

INTEGRAL TO SUM IDENTITY

$$\int_0^\infty dx\, x \left[j_{\ell+1}(xp_1)j_\ell(xp_2) + j_{\ell+1}(xp_2)j_\ell(xp_1)\right]$$

$$= \sum_n n\epsilon_n \sqrt{p_1 p_2}\left[j_{\ell+1}(np_1)j_\ell(np_2) + j_{\ell+1}(np_2)j_\ell(np_1)\right]$$

$$\epsilon_n = 1/2, \quad n = 0, \quad 1, \quad n > 0$$

Eigenfunction expansion is thus

$$\frac{1}{|\vec{p}_1 + \vec{p}_2|^2} = \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} n\epsilon_n \sum_{j=0}^{\ell} w_j^{\ell,1} \sum_{s=-j}^{j}\Big[\phi^{[2+]}_{n\ell js}(\vec{p}_1)\phi^{[2-]}_{n\ell js}(\vec{p}_2) + \phi^{[2+]}_{n\ell js}(\vec{p}_2)\phi^{[2-]}_{n\ell js}(\vec{p}_1)\Big]$$

$$\phi^{[2],+}_{n\ell js}(\vec{p}) = \sqrt{p}\, j_{\ell+1}(np)Y_{js}(\hat{p}), \quad \phi^{[2],-}_{n\ell js}(\vec{p}) = \sqrt{p}\, j_\ell(np)Y_{js}(\hat{p})$$

HOW ABOUT INVERSE FOURTH POWER?

Could use λ = 2 in $$\frac{1}{|\vec{p}_1+\vec{p}_2|^{2\lambda}} = \sum_{\ell=0}^{\infty} \frac{p_<^{\ell}}{p_>^{\ell+2\lambda}} C_\ell^{(\lambda)}(\hat{p}_1\cdot\hat{p}_2)$$

But would mean need difference of 4 in radial piece

No decoupling integral for that (and conjecture cannot find one given divergence props. of sBFs)

Instead:

$$\frac{1}{|\vec{p}_1+\vec{p}_2|^4} = \frac{1}{2p_1p_2}\frac{\partial}{\partial(\cos\theta_{12})}\left[\frac{1}{|\vec{p}_1+\vec{p}_2|^2}\right]$$

$$\frac{d}{dx}\left[C_\ell^{(\lambda)}(x)\right] = 2\lambda C_{\ell-1}^{(\lambda+1)}(x)$$

Leads to eigenfunction expansion for inverse 4th power

RETURNING TO OUR FULL PROBLEM

$$I_{ij}(k) = \sum_{\ell m} \frac{4\pi}{2\ell+1} \int \frac{d\Omega_k}{4\pi} \int \frac{d^3\vec{q}_1}{(2\pi)^3} \frac{d^3\vec{q}_2}{(2\pi)^3}$$

$$\times \frac{N_\ell^{[1]}(q_1) N_\ell^{[2]}(q_2) Y_{\ell m}(\hat{q}_1) Y^*_{\ell m}(\hat{q}_1)}{q_1^{2n_1} |\vec{k}+\vec{q}_1|^{2n_1'} q_2^{2n_2} |\vec{k}+\vec{q}_2|^{2n_2'}}$$

$$\times \frac{P_{\rm lin}(q_1) P_{\rm lin}(q_2)}{|\vec{q}_1+\vec{q}_2|^{2n_3} |\vec{k}+\vec{q}_1+\vec{q}_2|^{2n_3'}} P_{\rm lin}(|\vec{w}_{ij}|)$$

$$\vec{w}_{15} = \vec{k}, \quad \vec{w}_{24} = \vec{k}+\vec{q}_2, \quad \vec{w}_{33} = \vec{k}+\vec{q}_1+\vec{q}_2.$$

We have now factorized the problem terms, so can incorporate them as additional factors on terms in just q_1 and q_2

Just left with 3-D convolutions of sBFs X power spectrum X power laws X spherical harmonics, do angular part analytically → 1-D

SCHEMATICALLY . . .

$$I_{15} \to \sum \int r^2 dr\ f^{n}_{\ell\ell'\ell''}(N;N';r) f^{n'}_{LL'L''}(N;N';r)$$

$$f^{n}_{\ell\ell'\ell''}(N;N';r) = \int q^2 dq\ q^N P_{\rm lin}(q) j_\ell(Nq) j_{\ell'}(N'q) j_{\ell''}(qr)$$

1-D is much much better than 9-D

NEXT STEPS

Convergence

Implementation

Compare with other methods: Simonovic + 2017 power law approach; Fang & McEwen, Gebhardt & Jeong

N-body integrator?

SOLUTIONS IN TERMS OF INTEGRALS OF LINEAR FIELDS AGAINST KERNELS

$$\tilde{\delta}(\vec{k},\tau) = \sum_{n=1}^{\infty}(2\pi)^{-3n}\int d^3\vec{q}_1\cdots d^3\vec{q}_n(2\pi)^3\delta_{\rm D}^{[3]}\left(\vec{k}-\sum_{i=1}^{n}\vec{q}_i\right)$$

$$\times F_n^{(s)}(\vec{q}_i)\tilde{\delta}_{\rm lin}(\vec{q}_1,\tau)\cdots\tilde{\delta}_{\rm lin}(\vec{q}_n,\tau),$$

$$\tilde{\theta}(\vec{k},\tau) = -f(\tau)\mathcal{H}(\tau)\sum_{n=1}^{\infty}(2\pi)^{-3n}\int d^3\vec{q}_1\cdots d^3\vec{q}_n$$

$$\times(2\pi)^3\delta_{\rm D}^{[3]}\left(\vec{k}-\sum_{i=1}^{n}\vec{q}_i\right)G_n^{(s)}(\vec{q}_i)\tilde{\delta}_{\rm lin}(\vec{q}_1,\tau)\cdots\tilde{\delta}_{\rm lin}(\vec{q}_n,\tau),$$

$$F_n(\vec{q}_i) = \sum_{m=1}^{n-1} \frac{G_m(\vec{q}_{\le m})}{(2n+3)(n-1)} \Big[(2n+1)\frac{\vec{k}\cdot\vec{k}_1}{k_1^2} F_{n-m}(\vec{q}_{>m}) + \frac{k^2(\vec{k}_1\cdot\vec{k}_2)}{k_1^2 k_2^2} G_{n-m}(\vec{q}_{>m})\Big]$$

$$G_n(\vec{q}_i) = \sum_{m=1}^{n-1} \frac{G_m(\vec{q}_{\le m})}{(2n+3)(n-1)} \Big[3\frac{\vec{k}\cdot\vec{k}_1}{k_1^2} F_{n-m}(\vec{q}_{>m}) + n\frac{k^2(\vec{k}_1\cdot\vec{k}_2)}{k_1^2 k_2^2} G_{n-m}(\vec{q}_{>m})\Big],$$

Sum over *m* represents all possible splittings of n^{th} order term into 2 lower order terms (n-m) X m

Terms in 1/*k* come from inversion of nabla